Using arithmetic operation meanings to solve routine problems

The various meanings of the arithmetic operations can be described by arithmetic templates that provide students useful tools for making sense of events and circumstances of the real world. A template provides an explicit way of talking about meaning. It is a language-based model that can be expressed as a number sentence. Consider the following story.

Mary has 7 pencils in her desk. Her friend gives her another 5 pencils that Mary puts in her desk. When Mary counts the pencils in her desk she finds that there are 12 pencils.

The words paint a picture of combining pencils that can be described as a 'put together' action. Seven pencils are put together with 5 pencils resulting in 12 pencils in the desk. That action can be represented symbolically by the number sentence: '7 + 5 = 12'. The number sentence is a mathematical way of representing the action and contains all the pertinent information except for the units (in this case, pencils). But how does one know where to place the numbers? A template provides that information.

The 'put together' template for addition can be expressed as follows.

Starting STUFF + Incoming STUFF = All STUFF

This template indicates how the numbers are arranged in a 'put together' situation. The numbers are inserted into the three slots.

Syntax (the arrangement of symbols) and semantics (the meaning of symbols) are important matters in any language (English, Cree, etc.). Syntax and semantics are also important matters in mathematics and specifically in relation to the arithmetic templates. Each slot has a particular meaning attached to it. In the case of the 'put together' template above, the first slot (reading from left to right) represents the count of the starting stuff. The second slot represents the count of the incoming stuff. The third slot represents the count of the result of the combining action. If the order of information in the slots is changed, the number sentence represents a different situation (e. g. '2 + 3 = 5' means "starting with 2" while '3 + 2 = 5' means "starting with 3"). This is sort of like the distinction between "*Harry ate the carrot.*" and "*The carrot ate Harry.*"

Having three slots for inserting numerical information is a consistent feature of all the templates that describe the four arithmetic operations. Any one of the three slots can be the question of interest for a problem (for example: 23 + ? = 45). This means that the slot on the right hand side of the equal sign is not the only legitimate question of interest.

Four steps for routine problem solving

Four steps are useful when using templates to do routine problem solving. The first step is to identify the relevant arithmetic template(s) and use it/them as the basis for writing a number sentence(s) that corresponds to the action or event. The second step is to select an algorithm(s) for working out the answer. The third step is to perform the algorithm(s) either mentally, by using pencil and paper, or by using a calculator. The fourth step is to evaluate the solution to determine whether or not it is reasonable. An example follows.

> **Daniel was broke. He went to visit his grandmother who gave him some money. On the way home, he found $24.57 on the street. At home, he counted his money and found that he had $72.32. How much money did his grandmother give him?**

Step 1: Identify the arithmetic template.

The action in the story is 'put together'. An addition number sentence, ? + 24.57 = 72.32, best represents that action.

Step 2: Select the arithmetic to do.

There are two ways to proceed in this case: (1) select subtraction by first using the inverse relationship between addition and subtraction to transform the addition number sentence to the subtraction number sentence: 72.32 - 24.57 = ? or (2) select addition by leaving the number sentence as it is.

Step 3: Do the arithmetic.

If subtraction was selected then do it in some way (mentally, paper and pencil, etc.). If addition was selected, then mentally add up from 24.57 until you reach 72.32 (this is the method some store clerks use to give change). For example, add $.43 gets one to $25. Add $47 gets one to $72. Add $.32 gets one to $72.32. Adding $.43, $47, and $.32 is equivalent to adding $47.75, the amount that needs to be added on to $24.57 to obtain $72.32.

Step 4: Check the solution.

You may have made an error. For example, you could have identified the problem as one that involves subtraction and could have written the number sentence for it as; '? - 24.57 = 72.32'. Unfortunately, it may not be a simple matter to know whether the answer that you obtained is correct. The best that can be done is to see if the answer is reasonable. In this case, you could ask the following kind of question. Does it make sense that the number for the answer should be smaller than the money Daniel had when he arrived home? This places the issue where it should be - considering the answer in terms of understanding what the problem is about.

The following points should be considered when teaching students to use the four-step approach for routine problem solving.

- ☑ It is preferable, at least in the early grades, that a symbol such as an empty box, be used to indicate that a numerical value is to be found. An empty box expresses well the idea that a number sits inside it but we do not know what the number is. Once the number is figured out, it can be placed inside the box.
- ☑ Once students have learned a template well (e. g. the 'take away' template for subtraction), they should be presented with more complex problems in which they must be able to identify that template.
- ☑ Writing a number sentence to represent real actions or events can help students move from specific to general ways of thinking in mathematics. For example, it can help them make more sense of algebra in later years.

Example (involving 'take away' subtraction)

Mom gave Sue some money to buy a birthday gift for her brother. Sue went to Wentworths and bought a gift for $11. The change was $7. How much money did mom give Sue?

Comments

Because your elementary school learning experiences likely involved "do some arithmetic that hopefully works", you might want to do: **11 + 7 = $18**. Think about this though. Is the problem painted by the story about combining or removing? Money was not combined. It was removed by spending it and getting change. This situation is best modeled by 'take away' subtraction.

The 'take away' template is: **starting stuff - removed stuff = stuff left**.

Applying this template to the problem, the number sentence becomes: **# - 11 = 7**.

This number sentence can be rearranged to a convenient calculation form: **11 - 7 = #.** However, a grade 2/3 child would not likely see this number sentence as a starting point because it says that the story is about combining (when it is not). A grade 2/3 child can obtain the answer to **#** from **# - 11 = 7** by trying a number for **#** (e.g. 15, removing 11, and seeing if you get 7). This trial and error approach to obtaining the value for **#** supports the 'take away' meaning of subtraction involved in the story.

Example (involving 'groups of' multiplication)

> **John gave George 20 bags of marbles. John told George that there were 120 marbles in all and that each bag had the same number of marbles. How many marbles were in a bag?**

Comments

Because your elementary school learning experiences likely involved "do some arithmetic that hopefully works", you might want to do 120 ÷ 20 = 6. Think about this though. Is the problem painted by the story about splitting up into equal groups or has that splitting up been done already? There is no action of splitting up into equal groups in the story. The story paints a picture of a static situation (equal amounts already in a bunch of bags). This situation is best modeled by 'groups of' multiplication.

The 'groups of' template is: **number of groups x number in group = all (total)**.

Applying this template to the problem, the number sentence becomes: **20 x # = 120**.

This number sentence can be rearranged to a convenient calculation form: **120 ÷ 20 = #.** However, a grade 3/4 child would not likely see this number sentence as a starting point because it says that the story is about splitting up (when it is not). A grade 3/4 child can obtain the answer to **#** from **20 x # = 120** by trying a number for **#** (e.g. 4, multiplying by 20 or adding 20 four times, and seeing if you get 120). This trial and error approach to obtaining the value for **#** supports the 'groups of' meaning of multiplication involved in the story.

A template – a flexible tool for routine problem solving

Arithmetic templates are tools that students can use to solve problems. ***Once the tools are well understood***, it does not matter how they are applied to routine problem solving as problems can often be viewed in more than one way. The suggestion is to abandon "dogmatism" once students understand the templates. The only useful purpose served by dogmatism is to help sharpen ideas. Once that purpose is served, it is wise to abandon any rigid adherence to "correct" ways of solving problems. Students need to understand that templates are tools they can use, not rigid ways of solving problems.

Problems can often be viewed from more than one perspective. An example follows.

> **I want to weigh my cat (actually, measure its mass) but it will not sit still long enough on the bathroom scale. What can I do?**

Many people would solve the problem as follows. First, weigh yourself (say 80.4 kg). Then grab the cat and weigh yourself holding on to it (say 85.3 kg). Finally, subtract (85.3 - 80.4) t
o get the answer . There is a critical question about this. *What justifies subtracting to get the answer?* Think about the question for a moment before reading further.

- The ‘take away’ or the comparison meaning of subtraction can be used to justify subtracting. For educational mind expansion, you will find a third meaning of subtraction involved in the discussion below – the ‘change in two measurements’ meaning as well as an approach based on ‘put together’.

- The problem can be viewed in a ‘take away’ sense in two ways. It can be thought of as: “Begin with you and the cat, remove the cat, that leaves you.” The number sentence for this viewpoint is; ’85.3 - ? = 80.4'. To get the answer, that number sentence could be transformed to one that is more convenient for calculating the answer, namely, ’85.3 - 80.4 = ?’. The second way of viewing the problem in the ‘take away’ sense is: “Begin with the cat and you, remove you, the cat remains.” The number sentence for this is; ’85.3 - 80.4 = ?’, a form that is convenient for calculating the answer.

- The problem can be viewed as a comparison of two sets. The first set is the weight of you and the cat. The second set is your weight. The difference (the excess) tells us how much heavier the first set is when compared to the second. The number sentence that represents this viewpoint is, ’85.3 - 80.4 = ?’. This sentence is convenient for calculating the answer.

- The problem can be viewed in the ‘change in two measurements’ sense of subtraction. Weighing #2 (you and the cat) subtract weighing #1 (the cat) is the change in the two weight measurements. The number sentence for this is, ’85.3 - 80.4 = ?’. This sentence is convenient for calculating the answer.

- The problem can be viewed as an imagined ‘put together' and that is modeled by addition. Your weight is combined with the cat’s weight resulting in your weight and the cat’s weight together. The number sentence for this is; ’80.4 + ? = 85.3'. This is not convenient for calculating the answer. It can be transformed to the convenient form; ‘85.3 - 80.4 = ?’.

The understanding of templates as tools emerges gradually as students acquire a good knowledge of each template in rich but single focus contexts. These contexts involve only one interpretation with respect to applying a template - the one that sharply fits the problem. Once students can do this well, they should be ready to consider multiple interpretations of a problem.

A reason for having students do problems in a variety of ways is to help them understand that the arithmetic templates are tools that can be applied to real events or actions in more than one way. Students need to understand that a tool is a tool; one can use it according to what seems to make the most sense at the time.